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Inverse Aeroacoustic Problem for a Streamlined Body Part 1: Basic Formulation

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The inverse aeroacoustic problem associated with a flat-plate airfoil in unsteady compressible flow is formulated in terms of a Fredholm integral equation of the first kind. The feasibility of determining the unsteady pressure along the airfoil surface from the radiated acoustic signal is demonstrated. It is shown that the Hadamard conditions of existence and uniqueness of the inverse solution can be satisfied. The third Hadamard condition for the continuous dependence of the solution on the input acoustic data shows the problem to be ill-posed. The singular value decomposition method with regularization is used to treat the associated ill-conditioned algebraic problem. Discretization and collocation techniques are used to represent the unsteady pressure on the airfoil. Both methods give very accurate reconstructions when "perfect" input data are used. The collocation method requires only a small amount of input data either from the far field or the near field, thus making it more suitable for applications.

Nomenclature

A	= vector containing modified input data
a_1, a_2, a_3	= gust components
B	= vector of unknowns
C_p	= coefficient of pressure
c	= chord length
c_0, c_∞	= speed of sound
G	= modified Green's function
G_f	= free-space Green's function
$H_0^{(2)}$	= zeroth-order Hankel function of second kind
$\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3$	= unit direction vectors
K	= flow parameter defined as $\sqrt{[(k_1 M/\beta^2)^2 - (k_3/\beta)^2]}$
K_1	= flow parameter defined as $k_1 M/\beta^2$
k_1, k_2, k_3	= gust wavenumbers
M	= Mach number, U_∞/c_0
$\mathcal{M}$	= matrix
n	= normal derivative direction
P, p	= pressure
R	= length defined by $\lvert x - y \rvert$
r_f	= distance from origin to far-field point
s	= arc length
t	= time
U	= freestream velocity
$\mathcal{U}$	= matrix of left singular vectors
u	= unsteady velocity
u_i	= left singular vector
$\mathcal{V}$	= matrix of right singular vectors
v_i	= right singular vector
$x_1, x_2, x_3, y_1, y_2, y_3$	= spatial coordinates
α	= Tikhonov regularization parameter
β	= compressible-flow parameter, $\sqrt{(1 - M^2)}$
γ	= aerodynamic transformation angle
Δ	= jump
θ	= angular position of the far-field measurement location
Λ	= matrix of singular values
ν	= characteristic amplitude of unsteady disturbance
ρ	= density
σ_i	= singular values
ϕ, Φ	= variables used for potential-jump theory
ω	= frequency
∇	= gradient

Subscripts

0	= steady-state conditions
$+$	= function evaluated at $x_2 = +\epsilon, 0 < \epsilon \ll 1$
$-$	= function evaluated at $x_2 = -\epsilon, 0 < \epsilon \ll 1$
∞	= upstream conditions

Superscripts

t	= transpose
$'$	= time-dependent quantities
$\tilde{}$	= Prandtl–Glauert plane
$\bar{}$	= conjugate

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Introduction

TRADITIONALLY, in studying aerodynamically generated sound, one first identifies and quantifies the sources of sound and then calculates the scattered sound. This direct approach has been used to study sound generated by incident acoustic and vortical disturbances or turbulence interacting with a body as well as sound from jets.

Although the direct approach is very useful in determining the physical and engineering parameters that affect the radiated sound level, there are instances when it is of interest to consider the inverse problem. For example, if certain far-field sound requirements must be met, the inverse approach can be used to determine how best to satisfy them. In other instances, it may be desirable to use the inverse problem as a nonintrusive method for determining the unsteady pressure on an aircraft wing or a turbomachinery blade from its far-field acoustic signal.

In this paper, we develop an inverse approach for the aeroacoustic problem of a streamlined body in a subsonic mean flow. The radiated sound is generated as a result of either the oscillatory motion of the body or the interaction of incident acoustic or vortical waves with the body. Our development of the inverse approach mirrors the development of the direct problem. The early treatments of the direct problem considered the case of a flat-plate airfoil and used the assumption that the unsteady flow disturbances are small. This allows the governing equation to be linearized about the steady-state solution. (For more details, see the recent review article by Atassi.[1]) The linearization, in effect, uncouples the unsteady part of the flow from the mean flow, and brings about a significant simplification for the treatment of the linearized unsteady problem.

Similarly, to formulate and characterize the essential features associated with the inverse problem, we consider the case of a two-dimensional flat-plate airfoil in unsteady subsonic flow. In the frequency domain, the governing equation reduces to a Helmholtz equation for a transform of the pressure. In the inverse aeroacoustic problem, the Helmholtz equation must be solved for the airfoil surface pressure subject to a far-field boundary condition that prescribes the acoustic field. Green's theorem is used to cast the boundary-value problem in terms of a Fredholm integral equation of the first kind.

This paper demonstrates the feasibility of performing the aeroacoustic inversion for a flat-plate airfoil. To this end, we first discuss the Hadamard criteria for well-posed problems and then develop an inversion method. It is shown that both the Hadamard criteria[2] of existence and of uniqueness of a solution can be satisfied. Once the existence and uniqueness are established, an inversion method is developed using the singular value decomposition (SVD) with embedded regularization techniques. It is shown that this method gives "perfect" solutions when "perfect" input data are supplied.

Mathematical Formulation

We consider a flat-plate airfoil in a subsonic, inviscid, isentropic, uniform flow. Further, we assume that an unsteady disturbance is added to the flow through oscillatory motion of the airfoil or incident acoustic or vortical waves. Because the disturbances are assumed to be small, the governing equations of motion can be linearized about the uniform mean flow quantities to obtain the linearized Euler's equations. For oscillatory disturbances, it is always possible, without loss of generality, to consider a single harmonic component and to factor out the explicit time dependence. We nondimensionalize length with respect to the half chord $c/2$, velocity with respect to U_∞, time with respect to $c/2U_\infty$, and the unsteady pressure p' with respect to $a\rho_\infty U_\infty$, where a is related to the amplitude of the unsteady disturbance. The mathematical problem then can be reduced to a two-dimensional Helmholtz equation

$$(\tilde{\nabla}^2 + K^2)P = 0 \tag{1}$$

for a transformed pressure, $P(\tilde{x}_1, \tilde{x}_2) = p' \exp[-i(k_1 t + MK_1 \tilde{x}_1 - k_3\tilde{x}_3/\beta)]$, in the Prandtl–Glauert coordinate system.[3] The Prandtl–Glauert coordinates are defined by $\{\tilde{x}_1 = x_1, \tilde{x}_2 = \beta x_2, \tilde{x}_3 = \beta x_3\}$, and $k_1 = \omega c/2u_\infty$ is the reduced frequency.

To predict the sound scattered by a flat-plate airfoil in unsteady compressible flow, the Helmholtz equation (1) must be solved subject to a causality condition far from the airfoil and a boundary condition along the airfoil and its wake. Several methods have been used to solve this direct aeroacoustic boundary-value problem. Adamczyk and Brand[4] solved the direct aeroacoustic problem using an expansion of the solution to the Helmholtz equation in terms of Mathieu functions. However, as noted by Adamczyk in a later paper,[5] these functions are difficult to evaluate. In addition, this approach works only for the flat-plate geometry and cannot be generalized for real airfoils. An asymptotic theory, developed later by Martinez and Widnall,[6] provides an analytical expression for the far-field pressure in the limit of high frequency.

Recently, Atassi et al.,[3] using Green's theorem, offered a method that gives the far-field acoustic pressure in terms of the airfoil surface pressure.This method is simple and works for all disturbance frequencies. In this method, Green's theorem is applied to solutions of the Helmholtz equation in a domain D_+ between an outer contour C that encloses the airfoil and an inner contour that consists of the slit starting at the leading edge of the airfoil and extending along its wake on the $\tilde{x}_1$ axis to downstream infinity. The application of Green's theorem yields

$$P(\tilde{x}) = \frac{1}{2\pi}\int_{-1}^{\infty}\left[\Delta P(\tilde{y}_1)\frac{\partial G_f(\tilde{x}\,|\,\tilde{y})}{\partial \tilde{y}_2} - G_f(\tilde{x}\,|\,\tilde{y})\Delta\frac{\partial P(\tilde{y})}{\partial \tilde{y}_2}\right]\mathrm{d}\tilde{y}_1 - \frac{1}{2\pi}\int_C\left[P(\tilde{y})\frac{\partial G_f(\tilde{x}\,|\,\tilde{y})}{\partial \tilde{n}} - G_f(\tilde{x}\,|\,\tilde{y})\frac{\partial P(\tilde{y})}{\partial \tilde{n}}\right]\mathrm{d}\tilde{s} \tag{2}$$

where the free-space Green's function G_f is given by

$$G_f(\tilde{x}\,|\,\tilde{y}) = (i\pi/2)H_0^{(2)}(K|\tilde{x} - \tilde{y}|) \tag{3}$$

In general, we can write

$$P = P_i + P_s \tag{4}$$

where P_i represents the incident acoustic field and P_s is the radiated or scattered field. We note that P_i is continuous along the $\tilde{x}_1$ axis and that P_s is an odd function of $\tilde{x}_2$, continuous along the wake $\tilde{x}_1 > 1$, and such that $\partial P_s/\partial \tilde{x}_2$ is continuous along the entire $\tilde{x}_1$ axis. Therefore, for $\tilde{x} \in D_+$, the scattered field is given by

$$P_s(\tilde{x}) = \frac{1}{2\pi}\int_{-1}^{1}\Delta P(\tilde{y}_1)\frac{\partial G_f(\tilde{x}\,|\,\tilde{y})}{\partial \tilde{y}_2}\,\mathrm{d}\tilde{y}_1 \tag{5}$$

and the incident field by

$$P_i(\tilde{x}) = -\frac{1}{2\pi}\int_C\left[P(\tilde{y})\frac{\partial G_f(\tilde{x}\,|\,\tilde{y})}{\partial \tilde{n}} - G_f(\tilde{x}\,|\,\tilde{y})\frac{\partial P(\tilde{y})}{\partial \tilde{n}}\right]\mathrm{d}\tilde{s} \tag{6}$$

If there are no incident acoustic waves, then P_s is known in the far field and the inverse problem is governed by Eq. (5). On the other hand, when there is an incident acoustic field P_i, then we can calculate P_i from Eq. (6) and then calculate the scattered field $P_s = P - P_i$. Thus in all cases the problem can be formulated in terms of the Fredholm equation (5) of the first kind.

Finally, aeroacoustic applications are different from other applications, such as acoustic holography, for which researchers have considered inverse acoustic problems. The existence of a mean flow is the main distinction. Other distinctions between this application and inverse problems such as holography is that the body is assumed to be completely rigid. For unsteady aerodynamic applications, where the body is assumed to be rigid, the only frequency or frequencies in the system will be directly associated with the unsteady disturbance. For instance, if the airfoil is oscillating with a frequency ω, the pressure response of the airfoil also will be at ω and likewise the radiated sound also will be at that frequency. There are, however, more than a single wavelength associated with the problem. For incident convected vortical waves, the wavelength is $2\pi U_\infty/\omega$. For acoustic waves, the wavelength is affected by the mean flow velocity and thus varies from one direction to another. The maximum wavelength occurs for waves propagating in the mean flow direction downstream whereas the minimum wavelength occurs for upstream propagating waves.

Existence and Uniqueness

The inverse aeroacoustic problem of concern here is defined as determining the unsteady pressure on a flat-plate airfoil from its associated far-field acoustic signal. The existence criterion for this problem follows directly from the statement of the problem, i.e., the acoustic signal must be associated with the acoustic scattering from the plate for the inversion to have a solution. For instance, if there are multiple solid bodies near the flat plate, Eq. (5) must be modified to account for the reflected sound from these boundaries. Thus, the existence of a solution depends both on the far-field signal and the geometry.

Once existence of a solution is established, we can show that the solution to Eq. (5) is unique. Let there be two solutions to the inverse problem, $\Delta P_1 \neq \Delta P_2$, such that the acoustic field produced by both is identical on a surface Σ enclosing the airfoil. Here, ΔP represents the jump $P(\tilde{x}_1, 0+) - P(\tilde{x}_1, 0-)$. Hence, it can be written that

$$P(\tilde{x}) = \int_{-1}^{1} \Delta P_1(\tilde{y}_1) \frac{\partial G_f(\tilde{x} \mid \tilde{y})}{\partial \tilde{n}} \, d\tilde{y}_1 \tag{7}$$

$$= \int_{-1}^{1} \Delta P_2(\tilde{y}_1) \frac{\partial G_f(\tilde{x} \mid \tilde{y})}{\partial \tilde{n}} \, d\tilde{y}_1 \tag{8}$$

for $\tilde{x} \in \Sigma$. Subtracting Eq. (8) from Eq. (7) results in the following expression valid for $\tilde{x} \in \Sigma$:

$$\int_{-1}^{1} [\Delta P_1(\tilde{y}_1) - \Delta P_2(\tilde{y}_1)] \frac{\partial G_f(\tilde{x} \mid \tilde{y})}{\partial \tilde{n}} \, d\tilde{y}_1 = 0 \tag{9}$$

We now show that the far-field pattern of the radiating solution vanishes everywhere outside of Σ. To this end, we consider the region outside the surface, Σ. A Green's function can be calculated such that $G(\tilde{x} \mid \tilde{y}) = 0$ on the surface.[7] Therefore, for any point outside the surface, the pressure is found from

$$P(\tilde{x}) = \int_{\Sigma} P(\tilde{y}) \frac{\partial G(\tilde{x} \mid \tilde{y})}{\partial \tilde{n}} \, ds(\tilde{y}_1) \tag{10}$$

Since $P(\tilde{y})$ vanishes everywhere on Σ, it is clear that $P(\tilde{x}) = 0$ everywhere outside Σ.

Using a theorem established by Colton and Kress,[8] which states that if the radiating solution to the Helmholtz equation vanishes in the far field, then the radiating solution is zero everywhere outside of the body, we conclude that $P(\tilde{x})$ is zero everywhere outside of the airfoil.

Finally, we note that inverse solutions to double-layer potentials of the form

$$\Phi(\tilde{x}) = \int_{-1}^{1} \phi(\tilde{y}_1) \frac{\partial G_f(\tilde{x} \mid \tilde{y})}{\partial \tilde{n}} \, d\tilde{y}_1 \qquad \tilde{x} \in \Re^2/[-1, 1] \tag{11}$$

are not unique. However, these solutions may be rendered unique if we impose that the solution be bounded at the trailing edge of the airfoil, $\tilde{y}_1 = 1$. This condition corresponds to imposing the Kutta condition in thin-airfoil theory. Using potential jump theory, Eq. (11) becomes

$$\Phi_+(\tilde{x}_1) - \Phi_-(\tilde{x}_1) = \phi(\tilde{x}_1) = 0 \qquad \tilde{x}_1 \in [-1, 1] \tag{12}$$

For the problem at hand, $\phi(\tilde{x}_1) = \Delta P_1(\tilde{x}_1) - \Delta P_2(\tilde{x}_1)$ and, therefore,

$$\Delta P_1(\tilde{x}_1) = \Delta P_2(\tilde{x}_1) \tag{13}$$

Hence, the inverse aeroacoustic solution must be unique.

After establishing the condition for existence and uniqueness for the inverse aeroacoustic problem, we now consider the third Hadamard criterion,[2] which requires that the solution be continuously dependent on the data. In two dimensions, the relationship between the far-field pressure and the near-field pressure contains a factor of $1/\sqrt{(r_f)}$. Because of this dependence, errors on the order of $\delta p'$ in the far-field input data become errors on the order of $\sqrt{(r_f)}\delta p'$ in the near field. Hence, small changes in the input data can create large changes in the solution. This violates the third Hadamard criterion and shows that the inverse aeroacoustic problem will, in general, be ill-posed. For ill-posed problems where the third Hadamard criterion fails, solutions must be obtained using optimization or regularization techniques. The next sections outline the techniques used for solving the inverse aeroacoustic problem.

Method of Solution

General Techniques

The formulation of the inverse aeroacoustic problem, as well as many other inverse problems, leads to a Fredholm integral equation of the first kind, Eq. (5). A method often used for solving such an integral equation is to cast it in matrix form and then to solve the matrix equation. However, the matrix equation that represents a Fredholm integral equation is ill-conditioned and cannot be solved by standard inversion of the matrix. Instead, an optimization method like the singular value decomposition (SVD)[9] method must be used.

SVD is an optimization method for solving overdetermined and/or ill-conditioned matrices. Details about this method are given elsewhere.[10] Here we only include a brief outline of the SVD method.

In general, one needs to solve the matrix equation

$$[A] = [\mathcal{M}][B] \tag{14}$$

The matrix $\mathcal{M}$ of dimension $m \times n$ with $m \geq n$, can be represented as

$$\mathcal{M} = \mathcal{U}\Lambda\bar{\mathcal{V}}^t \tag{15}$$

where Λ is an $m \times n$ diagonal matrix with the first n diagonal terms containing the singular values, denoted here as σ_i, and the others containing zero; $\mathcal{U}$ is an $m \times m$ matrix containing the left singular vectors $\boldsymbol{u}_i$ in its columns; and $\mathcal{V}$ is an $n \times n$ matrix containing the right singular vectors $\boldsymbol{v}_i$ in its columns.

The singular values are the nonnegative square roots of the eigenvalues of the matrix

$$\bar{\mathcal{M}}^t\mathcal{M}$$

so that

$$\bar{\mathcal{M}}^t\mathcal{M}x_i = \sigma_i^2 x_i$$

and the singular vectors contained in the columns of $\mathcal{U}$ and $\mathcal{V}$ satisfy

$$\mathcal{M}\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i \tag{16}$$

$$\bar{\mathcal{M}}^t\boldsymbol{u}_i = \sigma_i \boldsymbol{v}_i \tag{17}$$

From this decomposition, one constructs the solution to Eq. (14) as follows:

$$[\boldsymbol{B}] = \sum_{\sigma_i \neq 0} \frac{[A] \cdot [\boldsymbol{u}_i]}{\sigma_i} [\boldsymbol{v}_i] \tag{18}$$

In effect, the vectors $\boldsymbol{u}_i$ form an orthonormal basis for the known field $\boldsymbol{A}$, and the vectors $\boldsymbol{v}_i$ form an orthonormal basis for the solution field $\boldsymbol{B}$. The solution is constructed using the basis vectors $\boldsymbol{v}_i$ with weighting constants given by $(\boldsymbol{A} \cdot \boldsymbol{u}_i)/\sigma_i$.

When a matrix is ill-conditioned, the singular values decay to zero quickly. To avoid dividing by these small values, regularization techniques must be embedded in the SVD method. One possible regularization technique is the spectral cutoff method. In this method, one sets the value of the smallest allowable singular value or cutoff. The solution then is constructed from only those basis functions corresponding to singular values greater than the cutoff. Given a cutoff value, or regularization parameter, α, the solution becomes

$$[\boldsymbol{B}] = \sum_{\sigma_i > \alpha} \frac{[A] \cdot [\boldsymbol{u}_i]}{\sigma_i} [\boldsymbol{v}_i] \tag{19}$$

Another regularization that can be used is known as Tikhonov regularization.[11] When applying the Tikhonov regularization scheme within the SVD, the singular values are simply modified by a damping factor α as follows:

$$[\boldsymbol{B}] = \sum_{i=1}^{n} \frac{\sigma_i}{\alpha + \alpha_i^2} ([A] \cdot [\boldsymbol{u}_i])[\boldsymbol{v}_i] \tag{20}$$

The Tikhonov regularization essentially damps out the influence of the basis vectors that correspond to the very small singular values. Implementation of the SVD method and the choice of the accompanying regularization parameter for the current problem will be discussed shortly. The SVD method has been used successfully in other related inverse acoustic problems.[12,13]

Solution Procedure

Before performing the SVD described in the preceding section, one must transform the integral equation (5) into a matrix equation. Since the acoustic signal is measured in the far field, we can consider either the full form of the kernel, i.e., the Hankel function, or its far-field expansion. In addition, the integration can be performed either for the variable y_1 from -1 to 1, where -1 corresponds to the leading edge of the airfoil and 1 corresponds to the trailing edge, or for the variable γ from 0 to π, where γ is defined as $y_1 = -\cos\gamma$. The variable γ is often used to avoid the singularity at the leading edge. Once a form of the integral equation is chosen, then the related matrix equation can be obtained. Two techniques for transforming the integral equation into a matrix equation were considered in this study. The first is a simple quadrature rule, and the second is a collocation method. Both produce ill-conditioned matrix equations, which then are solved using the SVD method. To implement the SVD method a regularization technique and a regularization parameter must be chosen.

A simple transverse gust case with $M = 0.4$ and $k_1 = 5.0$ serves as a test case for comparing the effect of the different schemes on the inverse solutions. Input data for testing the inversion method are provided by a semianalytical solution to the direct problem.[3] We solve the direct problem by using Possio's integral equation to calculate the pressure jump along the flat-plate airfoil and Eq. (5) to calculate the radiated sound. We define the far-field pressure at 79 equally spaced points on a circular arc located at a nondimensional radius of $r = 100$, from 0 to π but not including the endpoints, because adding these points will render the matrix $\mathcal{M}$ singular. The data from the direct problem are recorded with five-decimal-point accuracy. We defined this as "perfect" input data.

We have found a few schemes that provide "perfect" reconstructions of the pressure jump along the flat-plate airfoil, when "perfect" input data from the far field are used. A description of the development of these schemes follows.

A scheme for solving the inverse aeroacoustic problem consists of three steps: 1) specifying the form of the kernel of the integral equation and the integration parameter; 2) defining the method for transforming the integral equation into a matrix equation; and 3) setting the regularization. Several combinations of these three choices have been tested. We should note that for all of the schemes tested, the regularization is not really a free choice. Instead, calibration defines which of the above-mentioned regularization techniques should be used and the associated regularization parameter. The calibration consists of applying the two regularization techniques with several different regularization parameters to test cases with different Mach numbers and reduced frequencies. The combination that gives the best results for all tested combinations of Mach number and reduced frequency becomes the choice for that scheme. An example calibration is discussed shortly.

Quadrature Method

We first tested schemes that use a quadrature method for discretization of the integral equation. The difference between schemes in this set of tests depends on the form of the original integral equation. The very first scheme we tried used the exact form of Eq. (5). The matrix equation was formed with the trapezoidal quadrature rule. For this scheme, it was determined through calibration that the spectral cutoff regularization did not work and that the Tikhonov regularization scheme was best when $\alpha = 10^{-7}$. An example of the calibration can be seen in Fig. 1, which includes the real and imaginary parts of the reconstructed unsteady pressure jump $C_{\Delta p'}$. We nondimensionalize the unsteady pressure with respect to $a_2 \rho U_\infty$, where a_2 is the amplitude of the transverse gust. In Fig. 1, the leading edge of the airfoil is located at -1.0 on the airfoil axis and the trailing edge is located at 1.0. Here, the Tikhonov scheme is tested with several regularization parameters for a Mach number of 0.6 and a reduced frequency of 3.0. It shows that both $\alpha = 10^{-7}$ and $\alpha = 10^{-6}$ give the best reconstructions. However, $\alpha = 10^{-7}$ gives more accurate results for other choices of Mach number and reduced frequency that were tested but that are not shown here.

The reconstruction for our test case of $M = 0.4$ and $k_1 = 5.0$ produced by this first scheme when using "perfect" input data is shown in Fig. 2. The reconstruction is denoted case A and is represented by the dotted line. Again, the figure includes the real and imaginary parts of the nondimensional unsteady pressure jump. As in the calibration case, the reconstructions are not "perfect" even though "perfect" input data have been used, and, as expected, when the frequency $K = k_1 M/\beta^2$ decreases, the reconstructions become worse.

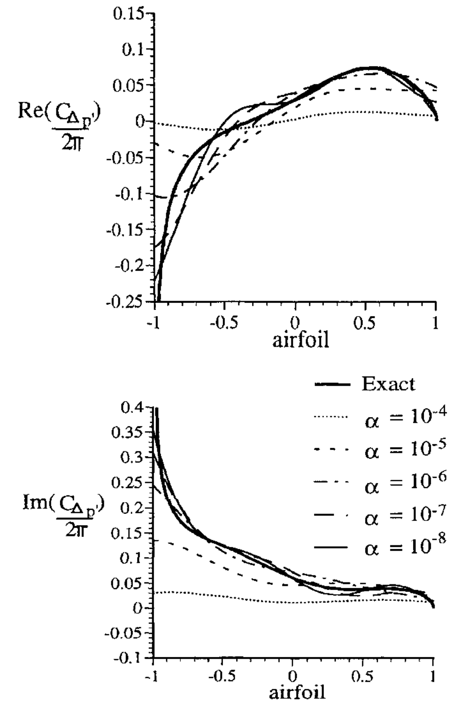

Fig. 1 Effect of varying the Tikhonov regularization parameter α for case A: real (top) and imaginary (bottom) parts of the unsteady surface pressure ($M = 0.6, k_1 = 3.0$).

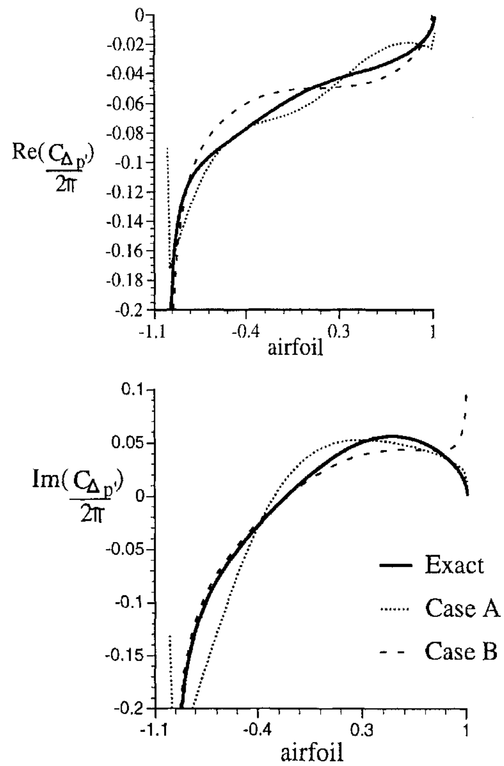

Fig. 2 Comparison of reconstructions using the quadrature methods: real (top) and imaginary (bottom) parts of the unsteady surface pressure ($M = 0.4, k_1 = 5.0$).

Most of the error in the reconstruction stems from the singular behavior of the unsteady pressure at the leading edge. Although this behavior is inherent in the problem, i.e., the information characterizes the kernel, computationally it is difficult to capture. In Fig. 2, case A, it can be seen that the magnitude of the unsteady pressure increases near the leading edge, but the true singular behavior is not captured. Instead, the computed solution suddenly decreases at the leading edge. This adds unwanted oscillations in the solution along the rest of the airfoil.

To aid the numerics in capturing the leading-edge singularity, the transformation $y_1 = -\cos\gamma$ is used in the original form of the integral equation. With this transformation, Eq. (5) becomes

$$P(\tilde{x}) = \frac{-iK\tilde{x}_2}{4}\int_0^\pi \Delta P(\gamma)\sin\gamma \frac{H_1^{(2)}(K|\tilde{x}+\cos\gamma\hat{i}|)}{|\tilde{x}+\cos\gamma\hat{i}|}\,\mathrm{d}\gamma \tag{21}$$

Solving the matrix equation for the quantity $\Delta P(\gamma)\sin\gamma$ ensures that the computed quantity is finite at the leading edge. Division by $\sin\gamma$ produces the singularity at the leading edge; however, it also may affect the behavior at the trailing edge. Thus the Kutta condition may not be satisfied rigorously. Any resulting singular behavior will be confined to a very small region near the airfoil trailing edge and will not influence the solution along the entire airfoil. Using 1) Eq. (21) as the form of the integral equation, 2) the trapezoidal rule to obtain the matrix equation, and 3) Tikhonov regularization with $\alpha = 10^{-6}$ in the SVD method, one obtains results for $M = 0.4$ and $k_1 = 5.0$ as shown by the dashed line in Fig. 2 and denoted case B.

Overall, the scheme using the transformation in the integral equation gives much better reconstructions, but still they are not "perfect." In particular, for small K, not shown here, the reconstructions are very poor. The reason for this difficulty can be traced to the relatively small dependence of the kernel phase on $\tilde{y}_1$. Indeed, for large $|\tilde{x}|$, $K|\tilde{x}-\tilde{y}|$ shows small variation as $\tilde{y}_1$ varies from -1 to 1. These variations become even smaller as K decreases. This is consistent with the fact that, as K becomes small, the unsteady surface pressure acts as a compact source and the far field does not depend on the details of the unsteady pressure distribution.

Thus an alternative method can be developed that uses the asymptotic behavior of the Hankel function to factor out the dependence on $|\tilde{x}|$. The integral equation (5) becomes

$$P(\tilde{x}) \sim \sqrt{\frac{K}{8\pi}}\frac{\exp[-i(K|\tilde{x}| - \pi/4)]}{|\tilde{x}|^{\frac{1}{2}}}\sin\tilde{\theta} \times \int_{-1}^{1}\Delta P(\tilde{y}_1)\exp(iK\tilde{y}_1\cos\tilde{\theta})\,\mathrm{d}\tilde{y}_1 + \mathcal{O}\left(\frac{1}{|\tilde{x}|^{\frac{3}{2}}}\right) \tag{22}$$

We introduce

$$f(\tilde{\theta}) = P(\tilde{x})\sqrt{\frac{8\pi}{K}}\exp[i(K|\tilde{x}| - \pi/4)]\frac{|\tilde{x}|^{\frac{1}{2}}}{\sin\tilde{\theta}} = \int_{-1}^{1}\Delta P(\tilde{y}_1)\exp(iK\cos\tilde{\theta}\tilde{y}_1)\,\mathrm{d}\tilde{y}_1 \tag{23}$$

which represents the integral equation to solve.

Again, we use the trapezoidal rule to discretize the matrix and the transformation $y_1 = -\cos\gamma$ to avoid the difficulty associated with the leading-edge singularity. Figure 3 shows the solution obtained from the SVD method with embedded Tikhonov regularization when "perfect" input data are supplied. The optimal regularization parameter for this method is $\alpha = 10^{-4}$. It is seen that these reconstructions are extremely accurate. This method gives "perfect" reconstructions at all values of K that have been tested. These values of K range from 0.01 to 20.0.

Although at this point we have demonstrated an inversion scheme that works well, there are two improvements that are necessary. First, the best scheme discussed thus far relies on the use of the asymptotic form of the kernel. When this form of the kernel is used, the input data must come from the far field. However, for application, one would like the flexibility to use far-field or near-field information. Second, thus far, a simple quadrature rule has been used to discretize the integral equation. It has been determined that at least 20 points on the airfoil are necessary for accurate discretization. Hence, at least 20 input points must be available. In application, this requirement may be too high.

Using either a higher-order quadrature method or a collocation method to form the matrix equation, as opposed to the trapezoidal quadrature rule, can lessen the input data requirement. Because the

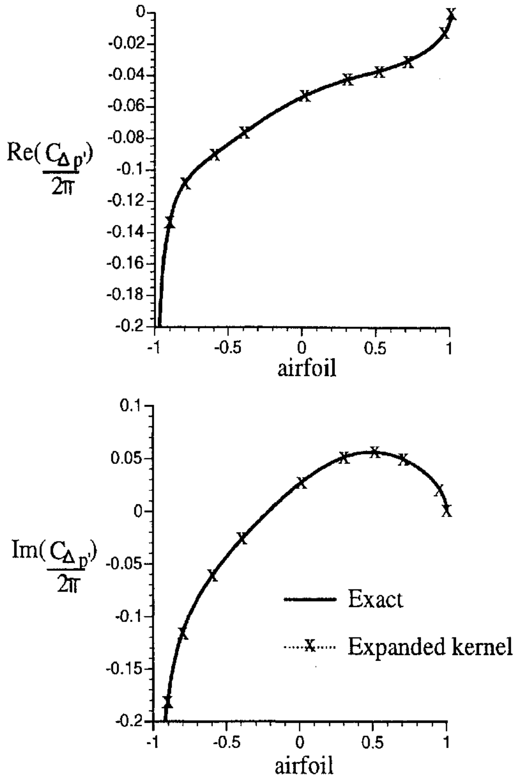

Fig. 3 Reconstruction using the quadrature method with expanded kernel: real (top) and imaginary (bottom) parts of the unsteady surface pressure ($M = 0.4, k_1 = 5.0$).

matrix is more easily defined using a collocation technique as compared to using a higher-order quadrature scheme, and, because a collocation series can be easily chosen to include the basic features of the unsteady pressure, we chose to test the collocation method.

Collocation Method

Since the unsteady pressure on the airfoil represents an aerodynamic response, some information regarding its nature is already known from the direct problem; the unsteady pressure is characterized by a square-root singularity at the leading edge and a Kutta condition at the trailing edge. These characteristics must be inherent in an inverse solution and hence the collocation series is chosen to include these features:

$$\Delta P(\gamma) = A_0\cot\left(\frac{\gamma}{2}\right) + \sum_{n=1}^{\infty}A_n\sin n\gamma \tag{24}$$

where $\tilde{y}_1 = -\cos\gamma$. Just a few terms of this series provide a good representation of the unsteady pressure jump along the flat-plate airfoil even when the airfoil forces are noncompact. This implies that only a few input points are needed. Moreover, this series can be substituted into the integral equation with the kernel in either its general form or its asymptotic form. This adds flexibility in the sense that the input data can be obtained either in the midfield or in the far field. When the collocation series is substituted into the asymptotic form of the integral equation, the integration can be carried out analytically. Hence, for convenience, we only consider the asymptotic form of the integral equation. Substituting Eq. (24) into Eq. (23) and integrating gives

$$f(\tilde{\theta}) = A_0\pi[J_0(K\cos\tilde{\theta}) - iJ_1(K\cos\tilde{\theta})] + \sum_{n=1}^{\infty}A_n\pi(-i)^{n-1}n\frac{J_n(K\cos\tilde{\theta})}{K\cos\tilde{\theta}} \tag{25}$$

where J_n is the Bessel function of order n. The Bessel functions have the property that

$$J_n(z) \sim (1/\sqrt{2\pi z})(ez/2n)^n \quad \text{as} \quad n \to \infty \tag{26}$$

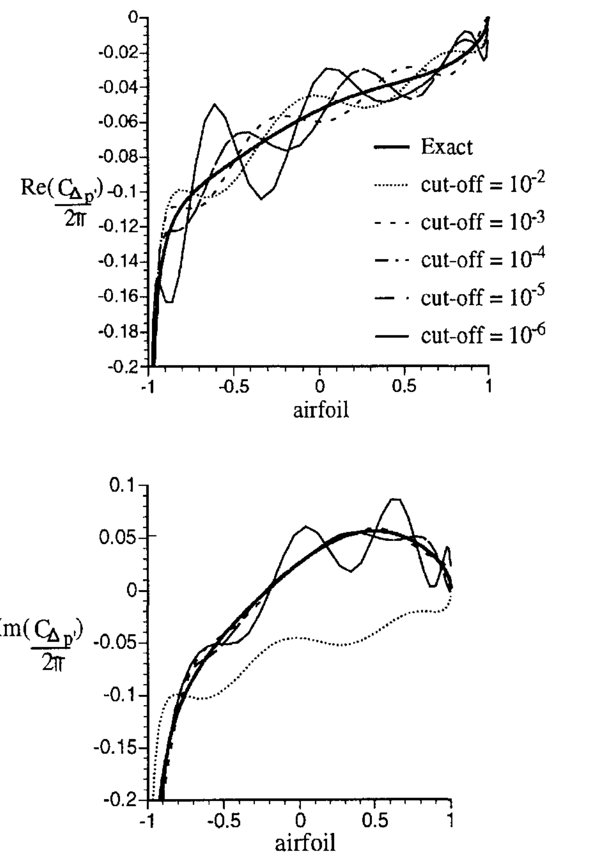

Fig. 4 Effect of varying the cutoff parameter with collocation technique: real (top) and imaginary (bottom) parts of the unsteady surface pressure (M = 0.4, k_1 = 5.0).

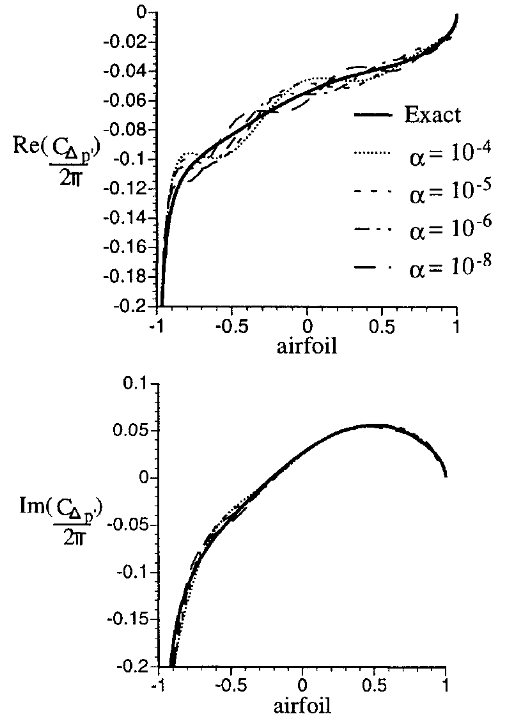

Fig. 5 Effect of varying the Tikhonov parameter with collocation technique: real (top) and imaginary (bottom) parts of the unsteady surface pressure (M = 0.4, k_1 = 5.0).

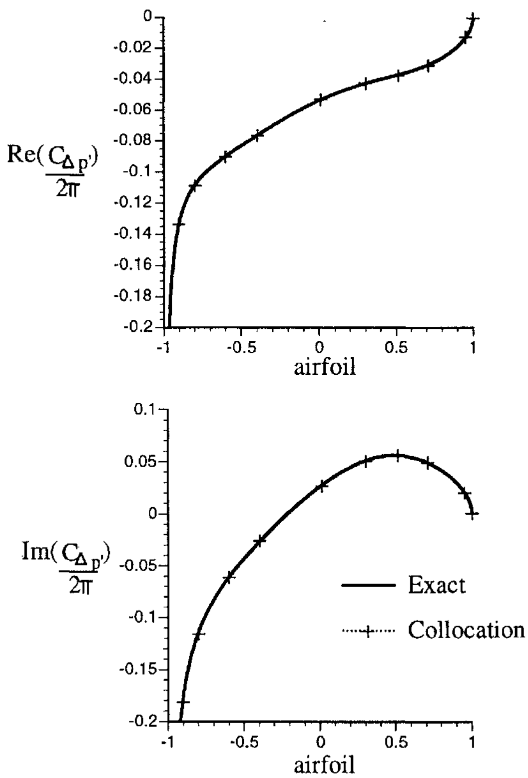

Fig. 6 Reconstruction using a priori regularization with collocation technique: real (top) and imaginary (bottom) parts of the unsteady surface pressure (M = 0.4, k_1 = 5.0).

From this, it is clear that the values of the elements in the columns of $\mathcal{M}$ decrease as the column number increases, thus leading to an ill-conditioned matrix equation. It is also clear that, as K increases, the columns decay to zero more slowly. It follows that, as the airfoil radiation problem becomes more noncompact, which is measured by the value of K, more terms can be used in the series representation without ill-conditioning the matrix.

Solving the matrix equation using the SVD method with the basic regularization schemes mentioned previously, one never obtains a "perfect" reconstruction. Some attempts of calibration for the spectral cutoff method and the Tikhonov method are shown in Figs. 4 and 5, respectively. This result is a peculiarity of the SVD method, and it is discussed further in the next section.

To ensure a "perfect" reconstruction when "perfect" input data are used, the implementation of the regularization must be altered. The alteration follows from the behavior of Eq. (25). As more terms are used in the series, the matrix equation becomes more ill-conditioned. However, the number of terms in the series can be limited such that the condition number of the matrix is close to 1. In other words, by choosing the correct number of terms in the series, the matrix remains well conditioned. Truncating the series is a form of regularization, and we term this the a priori cutoff method.

This a priori regularization allows for direct inversion of the matrix. If the series is limited, however, the system of equations will most likely be overdetermined, which again hinders direct inversion. The SVD method can again be used to solve the overdetermined system. In this case, because the regularization is done a priori, an embedded regularization is not necessary.

The a priori cutoff, i.e., number of terms used in the collocation series, is chosen based on the singular values of the matrix. The restriction is that these singular values all must be greater than the cutoff parameter 0.01, found through calibration using "perfect" input data. This allows for "perfect" reconstruction for all cases of K, where $0.01 < K < 20$. Figure 6 shows the reconstruction using this a priori cutoff technique for the case $M = 0.4$, $k_1 = 5.0$.

Analysis of the SVD Method

The order in which the schemes have been covered here shows the progression toward an accurate and flexible solution method for the inverse aeroacoustic problem. The first two schemes, cases A and B, did not give very accurate results. Case B was an improvement over case A due to special treatment of the leading-edge singularity. It was then shown that when case B was modified to incorporate

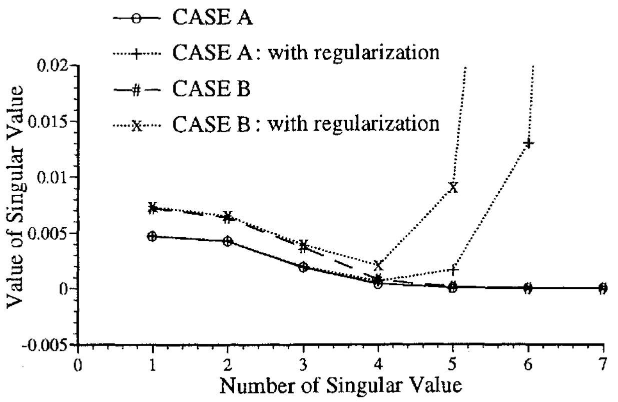

Fig. 7 Singular values for quadrature methods ($M = 0.4, k_1 = 5.0$).

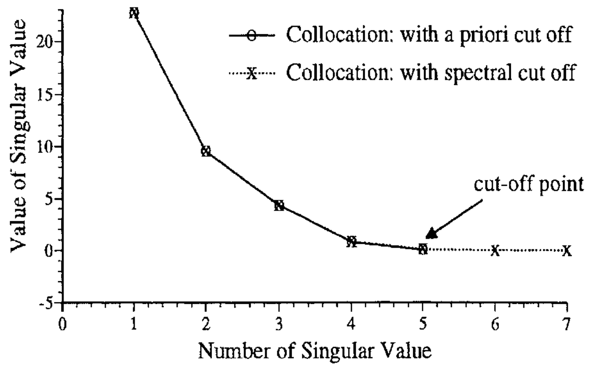

Fig. 8 Singular values for collocation method ($M = 0.4, k_1 = 5.0$).

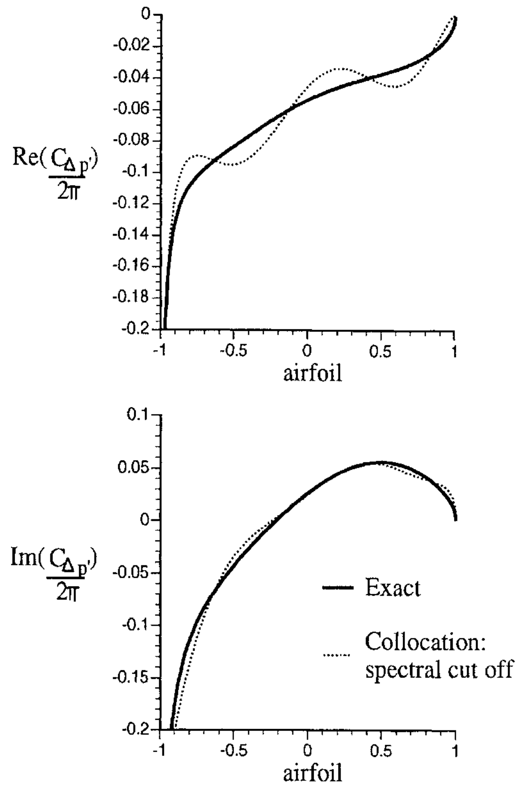

Fig. 9 Collocation method with spectral cutoff embedded in SVD: real (top) and imaginary (bottom) parts of the unsteady surface pressure ($M = 0.4, k_1 = 5.0$).

an expanded kernel, the inversion was "perfect." This asymptotic form of the kernel eliminated difficulties associated with using far-field input data. The collocation method was presented as a more flexible method that still produces very accurate reconstructions. This progression to "perfect" solutions could have been predicted to some degree by studying the singular values of the corresponding matrix equations.

The solution is constructed in the SVD method as follows:

$$[B] = \sum_i \frac{[A] \cdot [u_i]}{\sigma_i} [v_i] \tag{27}$$

Now consider two different forms of the matrix equation arising from two different schemes. One gives rise to small singular values and one to large singular values. If both sets of singular values are to be used in Eq. (27) with the fixed input vector A to determine B, then for the set of small singular values, the singular vectors u_i or v_i or both must also contain small numbers as compared to the singular vectors corresponding to the larger set of singular values. When computing the solutions, the smaller numbers are more susceptible to numerical error. So it is expected that the form of the matrix equations giving rise to the largest singular values would be the more accurate formulation.

The singular values for cases A and B are plotted in Fig. 7. There are two curves for each case. The first curve shows the actual singular values, whereas the second curve shows the singular values after regularization. From the plot of the regularized singular values, it is easy to see how the Tikhonov regularization acts to damp out the influence of the basis vectors corresponding to the extremely small singular values. Overall, the singular values in case B are larger than the singular values for case A. For the expanded kernel case, not shown, the largest singular value is approximately 2.0 as compared to 0.007 for case B and 0.005 for case A. Finally, Fig. 8 shows that the largest singular value for the collocation method is approximately 23. As predicted, the greater the value of the largest singular value for a given scheme, the better the reconstruction.

A very peculiar problem associated with the SVD scheme occurs when the matrix is formed using the collocation method. It was mentioned earlier that using the a priori cutoff leads to "perfect" solutions. If, however, the matrix is allowed to become ill-conditioned, i.e., the number of terms in the series is not limited, then a "perfect" reconstruction cannot be obtained. This is true even when the spectral cutoff method is applied within the SVD method with the same value for the cutoff as is used in the a priori scheme that gives "perfect" results. Figure 9 shows the results obtained when seven terms are used in the series to produce the matrix instead of five, but the spectral cutoff method is used within the SVD method to ensure that only five singular values are actually used to reconstruct the solution. The results are not good. No improvement is seen when all seven of the singular values are used. This must be due to the fact that the SVD is really a least-squares type of algorithm. When it is used to make the best fit with information that may be redundant or even erroneous, it may yield inaccurate results.

Conclusion

The feasibility of determining the unsteady surface pressure along a streamlined body from the radiated sound is demonstrated. Theoretically, the unsteady pressure can be produced by a number of disturbances including aeroelastic vibration of the body, an incident acoustic wave, or an incident vortical gust. The three Hadamard conditions for well-posedness have been considered. It is shown that the first two criteria, existence and uniqueness of a solution, can be satisfied for this problem. However, the third condition, of continuous dependence of the solution on the input data, fails, thus rendering the problem ill-posed. Still, however, the inversion is feasible.

To determine the unsteady pressure on the airfoil from the radiated sound, a Fredholm integral equation of the first kind must be solved. Transforming this integral equation into a matrix equation, in general, leads to an ill-conditioned system. The matrix equation can be solved using the SVD method with added regularization. The schemes developed for solving this integral equation consist of three stages. In the first stage, one defines the form of the integral equation. Then one chooses a method for transforming the integral equation into a matrix equation. Finally, the type of regularization and the regularization parameter to use within the SVD method are set.

The schemes are tested for the case of a transverse vortical gust impinging on a flat-plate airfoil. Three schemes were found that give "perfect" reconstructions of the unsteady pressure on the airfoil when "perfect" input data are supplied.

It was shown that a scheme using the trapezoidal quadrature to transform the integral equation into a matrix equation could lead to "perfect" reconstructions. The form of the integral equation had to include a transformation to aid in capturing the leading-edge singularity in the unsteady pressure jump; in addition, the kernel had to be written in expanded form. For this scheme, the Tikhonov regularization was used.

The collocation method for transforming the integral equation to a matrix equation leads to two schemes that produce "perfect" reconstructions. These schemes also have the leading-edge singularity embedded. The difference between the two schemes is in the form of the kernel. The collocation technique works with both the general form and the expanded form of the integral equation. To obtain "perfect" reconstructions, the normal regularization techniques could not be used; instead, an a priori cutoff of the number of terms in the collocation series must be used.

The ability to find accurate solutions when "perfect" input data are available proves the feasibility of the inverse aeroacoustic problem. However, in application, "perfect" input data, as used in this study, will not exist. Instead, the challenge is to determine how to obtain accurate solutions when the input data are noisy. This is addressed in a separate paper.

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